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Graph Z_n and some graphs related to Z_n are determined by their spectrum

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Abstract

It is proved that graph Z_n is determined by its adjacency spectrum as well as its Laplacian spectrum; $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ is determined by its adjacency spectrum, where $n_1, n_2, \ldots, n_k$ are integers at least 2; W_n is not determined by its adjacency spectrum but is determined by its Laplacian spectrum; kZ_n, T_n are determined by their Laplacian spectrum, respectively, where k is a positive integer.

AMS classification: 05C50

Keywords: Spectrum of a graph; Cospectral graphs; Eigenvalues; Starlike trees

1. Introduction

We consider undirected graphs having no loops or parallel edges. All notions on graphs that are not defined here can be found in [1].

Let G be a graph with n vertices, $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. We assume $V(G) \neq \emptyset$ (and so $n > 0$). Let matrix $A(G)$ be the adjacency matrix of G, $d_G(v)$ be the degree of vertex v in G, and $D(G)$ be

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the diagonal matrix with degrees of the corresponding vertices of G on the main diagonal. Matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G. Denote the characteristic polynomial of the adjacency matrix $A(G)$ (Laplacian matrix $L(G)$) by $P_{A(G)}(\lambda)$ ($P_{L(G)}(\mu)$). The eigenvalues of $A(G)$ ($L(G)$) and the spectrum (which consists of eigenvalues) of $A(G)$ ($L(G)$) are also called the adjacency (Laplacian) eigenvalues of G and the adjacency (Laplacian) spectrum of G. Since both matrices $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are all real numbers. So we can assume that $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G)(= 0)$ are the adjacency eigenvalues and the Laplacian eigenvalues of G, respectively.

About the background of the question "which graphs are determined by their spectrum?", we refer to [3]. It seems hard to prove a graph to be determined by its spectrum. Only few graphs have been proved to be determined by their spectrum.

The following known results can be found in [3,4]:

(i) Graphs with the number of vertices less than 5, the path with n vertices P_n, the complete graph K_n, the regular complete bipartite graph $K_{m,m}$, the cycle C_n and their complements, the disjoint union of k disjoint paths $P_{n_1} + P_{n_2} + \cdots + P_{n_k}$ are determined by their spectrum with respect to the adjacency matrix as well as the Laplacian matrix.

(ii) The disjoint union of k complete graph, $K_{n_1} + K_{n_2} + \cdots + K_{n_k}$, is determined by their adjacency spectrum.

Remark. If we view an isolated vertex as P_1, the result 'the disjoint union of k disjoint paths is determined by its adjacency spectrum' would be wrong. For example, $P_7 + P_1$ is cospectral with $Z_3 + P_3$ with respect to the adjacency matrix (Z_3 is a tree defined in the following). The result holds only for all integers $n_1, \ldots, n_k$ greater than 1. For convenience, we refer an isolated vertex as K_1 not P_1 in this paper.

The following question is proposed in [3]: which trees are determined by their spectrum? We still do not know the answer. In this paper, three special graphs are involved. The following three graphs were denoted by Z_n ([2], p. 77), T_n and W_n, respectively (Fig. 1).

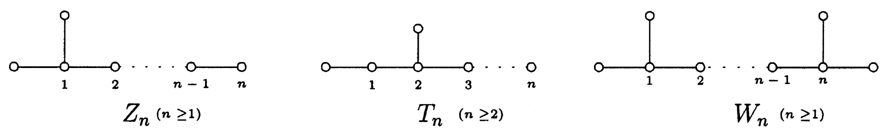

Fig. 1. Three special graphs.

Clearly, graph Z_n, T_n are trees with $n+2$ vertices and $n+1$ edges, respectively. W_n is a tree with $n+4$ vertices and $n+3$ edges.

This paper is constructed as following: In Section 2, we will prove that Z_n is determined by its adjacency spectrum and get a more general result. In Section 3, graphs Z_n, kZ_n, T_n and W_n will be proved to be determined by their Laplacian spectrum, respectively, where k is a positive integer.

2. Z_n is determined by its adjacency spectrum

The following lemmas will be frequently used throughout this paper.

Lemma 2.1 [3]. *For $n \times n$ matrices A and B, the following are equivalent*:

(i) *A and B are cospectral*;
(ii) *A and B have the same characteristic polynomial*;
(iii) $\mathrm{tr}(A^i) = \mathrm{tr}(B^i)$ *for* $i = 1, 2, \ldots, n$.

If A is the adjacency matrix of a graph, then $\mathrm{tr}(A^i)$ gives the total number of closed walks of length i. So cospectral graphs have the same number of closed walks of a given length i. In particular, they have the same number of edges (take $i = 2$) and triangles (take $i = 3$).

Lemma 2.2 [3]. *In a graph without* 4 *cycles, the number of closed walks of length* 4 *equals twice the number of edges plus four times the number of induced paths of length* 2.

Lemma 2.3 [5]. *Let Y be a subgraph of X, then $\lambda_{\max}(Y) \leqslant \lambda_{\max}(X)$. Furthermore, when Y is a proper subgraph, equality can hold only when X is not connected.*

Other useful tool is the following statement.

A tree in which exactly one vertex has degree greater than 2 is said to be starlike (see [11]). For starlike trees, we have:

Lemma 2.4 [8]. *No two non-isomorphic starlike trees are cospectral with respect to their adjacency matrices.*

Since the adjacency spectrum of W_n is the union of the spectra of the circuit C_4 and the path P_n ([2], p. 77), then the largest eigenvalue of W_n is 2 and W_n cannot be determined by its adjacency spectrum.

Now we prove our first result:

Theorem 2.5. *Graph Z_n is determined by its adjacency spectrum.*

Proof. The adjacency eigenvalues of Z_n are 0, $2\cos\frac{(2i+1)\pi}{2(n+1)}$, $i = 0, 1, \ldots, n$ ([2], p. 77). It gives $\lambda_1(Z_n) < 2$. For $n = 1$, graph Z_n is P_3 (the path with three vertices), it is determined by its spectrum. The result holds. For $n > 1$, suppose a graph T is cospectral with Z_n with respect to the adjacency spectrum. By Lemma 2.1, T is a graph with $n + 2$ vertices and $n + 1$ edges. Since the circuit has an eigenvalue 2, it cannot be an induced subgraph of T because of Lemma 2.3. Therefore T is a tree. Similarly, the star $K_{1,4}$ has an eigenvalue 2, so $K_{1,4}$ is not a subgraph of T. Also graph W_n has an eigenvalue 2, so T is a tree without any vertex of degree at least 4 and at most one vertex of degree 3. Since the path is determined by its spectrum, T is not a path. Therefore T is a starlike tree with the largest vertex degree 3. By Lemma 2.4, T is isomorphic to Z_n. □

We denote the disjoint union of k graphs $Z_{n_1}, Z_{n_2}, \ldots, Z_{n_k}$ by $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$; denote the disjoint union of k disjoint graphs Z_n by kZ_n; denote the following three graphs by G_1, G_2, G_3, which share a common property with their largest adjacency eigenvalues less than 2.

By using Maple, we find the characteristic polynomial $P_{A(G_2)}(\lambda)$ is a factor of the characteristic polynomial $P_{A(Z_8)}(\lambda)$. Hence the adjacency eigenvalues of G_2 is a part of that of Z_8. Thus we get the adjacency spectrum of G_2 is

$$2\cos\frac{\pi}{18}, 2\cos\frac{5\pi}{18}, 2\cos\frac{7\pi}{18}, 0, 2\cos\frac{11\pi}{18}, 2\cos\frac{13\pi}{18}, 2\cos\frac{17\pi}{18}.$$

Similarly, G_3 has eigenvalues

$$2\cos\frac{\pi}{30}, 2\cos\frac{7\pi}{30}, 2\cos\frac{11\pi}{30}, 2\cos\frac{13\pi}{30}, 2\cos\frac{17\pi}{30}, 2\cos\frac{19\pi}{30},$$
$$2\cos\frac{23\pi}{30}, 2\cos\frac{29\pi}{30}.$$

Similar to the proof of Theorem 2.5, we get:

Corollary 2.6. *Graphs G_1, G_2, G_3 are determined by their adjacency spectrum, respectively.*

For $n = 3$, T_n is Z_3; for $n = 4$, T_n is G_1; for $n = 5$, T_n is G_2; for $n = 6$, T_n is G_3. Theorem 2.5 and Corollary 2.6 imply that T_n is determined by its adjacency spectrum for $n < 7$. For $n = 7$, T_n is G_6 (see Fig. 3), from the spectra which displayed in [2] from p. 272 to p. 306, we know that it is also determined by its adjacency spectrum. But for $n > 7$, we do not know the answer with our skills.

The following can be deduced directly from their spectrum:

Corollary 2.7

(i) *$Z_8 + K_1$ is cospectral with $G_2 + Z_2$ with respect to the adjacency matrix*;
(ii) *$Z_{14} + 2K_1$ is cospectral with $G_3 + Z_4 + Z_2$ with respect to the adjacency matrix*;

(iii) *More generally*, $kZ_{14} + lZ_8 + (2k + l)K_1$ *is cospectral with* $kG_3 + lG_2 + kZ_4 + (k + l)Z_2$ *with respect to the adjacency matrix*, *where* k, l *are positive integers*.

The following gives a more general result.

Theorem 2.8. *Graph* $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ *is determined by its adjacency spectrum*, *where all* $n_1, n_2, \ldots, n_k$ *are greater than* 1.

Proof. Suppose a graph G is cospectral with $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ with respect to the adjacency matrix. Similar to the proof of Theorem 2.5, we find G and $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ have the same number of vertices, edges, and closed walks of length 4. At the same time, G is a forest with k components, and each component has no vertex of degree at least 4 and at most one vertex of degree 3.

First, we declare that G has no path component and the possible components of G are G_2, or G_3 (see Fig. 2), or Z_m $(m > 1)$, or K_1. Assume that there exist t $(t \geqslant 1)$ path components in G, then the number of induced paths of length 2 in G is less than that in $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ by t. By Lemma 2.2, the number of closed walks of length 4 of G is clearly less than that of $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$. So no path is a component of G. Therefore each component of G contains exactly one vertex of degree 3. Furthermore, G_1 (see Fig. 2) is not a component of G because the spectrum of G_1 contains an eigenvalue 1 ([2], p. 276), which is not an eigenvalue of $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$. Since the following three graphs (see Fig. 3) all have the largest eigenvalue 2 ([2], p. 276), then G_4, G_5, G_6 cannot be induced subgraphs of G by Lemma 2.3. So the possible components of G are G_2, G_3 (see Fig. 2), Z_m $(m > 1)$ or K_1. From their adjacency spectrum, we know that it is impossible that all components of G are G_2, or G_3, or K_1.

Second, we prove that G_2, G_3, K_1 are also not components of G. Suppose G is $aG_2 + bG_3 + Z_{m_1} + \cdots + Z_{m_l} + cK_1$, where $a + b + c + l = k$, a, b, l are positive integers, c is a nonnegative integer. We declare that each component of $Z_{m_1} + \cdots + Z_{m_l}$ is just one component of $Z_{n_1} + \cdots + Z_{n_k}$. Assume that there are t $(0 \leqslant t < l)$ pair-wise isomorphic components between $Z_{m_1} + \cdots + Z_{m_l}$ and $Z_{n_1} + \cdots + Z_{n_k}$. Deleting these t pair-wise isomorphic components simultaneously from G and $Z_{n_1} + \cdots + Z_{n_k}$, and denote the remaining by S and S', respectively. Obviously, S and S' are cospectral. So there are x $(= k - a - b - c - t)$ components (say $Z_{m_{i_1}}$,

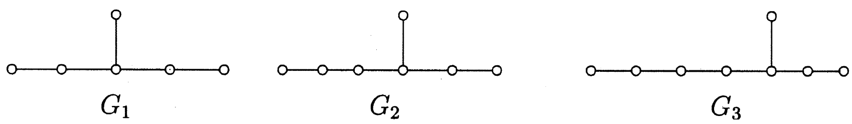

Fig. 2. Three graphs with their largest adjacency eigenvalue less than 2.

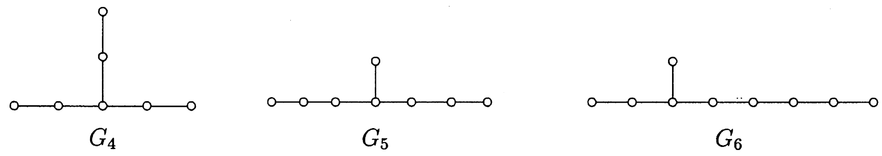

Fig. 3. Three graphs with their largest adjacency eigenvalue 2.

$\dots, Z_{m_{i_x}}$) in $Z_{m_1} + \cdots + Z_{m_l}$ which are not isomorphic to any components of S', and assume Z_n is one of these x components with the largest size. Then the spectrum of Z_n is part (not all) of the spectrum of some component(s) of S'. Since $2\cos\frac{\pi}{2(n+1)}$ is an eigenvalue of Z_n, there are at least one eigenvalue $\lambda = 2\cos\frac{\pi}{2(n+1)(2m+1)}$ (m is some positive integer) in the spectrum of S' and S, respectively. Since Z_n has the largest size among $Z_{m_{i_1}}, \dots, Z_{m_{i_x}}$, it follows λ is an eigenvalue of G_2, or G_3.

If λ is an eigenvalue of G_3, then the equality $\lambda = 2\cos\frac{\pi}{30}$ holds. It implies $n = 2$, $m = 2$ or $n = 4$, $m = 1$. If $n = 2$, then each component of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$ is Z_2. So both S' and S have the largest eigenvalue $2\cos\frac{\pi}{30}$ with multiplicity b and another eigenvalue $2\cos\frac{\pi}{18}$ (which is the largest eigenvalue of G_2) with multiplicity a. It follows that there are exactly a components Z_8 and b components Z_{14} in S'. So $2\cos\frac{\pi}{10}$ is an eigenvalue of S', but it is not an eigenvalue of S. A contradiction! If $n = 4$, the possible components of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$ are Z_2, or Z_3, orZ_4. Similar to the proof above, it is impossible for all components of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$ are Z_2 (or Z_4). Assume that Z_3 is a component of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$, since $2\cos\frac{\pi}{8}$ is an eigenvalue of Z_3, then $2\cos\frac{\pi}{8(2m'+1)}$ (m' is some positive integer) is an eigenvalue of S' for the same reason above. But it is never an eigenvalue of Z_8 nor Z_{14}. So Z_3 is not a component of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$. Hence both Z_2 and Z_4 are components of $Z_{m_{i_1}} + \cdots + Z_{m_{i_x}}$. It follows that there are exactly a components Z_8 and b components Z_{14} in S'. Since the x components of $Z_{m_1} + \cdots + Z_{m_l}$ are not isomorphic to any component of S', it forces the remaining $x + c$ components in S' are K_1. It contradict our hypothesis! Similarly, λ is also not an eigenvalue of G_2. Thus each component of $Z_{m_1} + \cdots + Z_{m_l}$ is just one component of $Z_{n_1} + \cdots + Z_{n_k}$. It follows that $aG_2 + bG_3 + cK_1$ cospectral with the remaining of $Z_{n_1} + \cdots + Z_{n_k}$ by deleting $Z_{m_1} + \cdots + Z_{m_l}$. But from their spectrum, we know it is impossible. Similarly, the graph G is neither the form $aG_2 + Z_{m_1} + \cdots + Z_{m_l} + cK_1$ nor the form $bG_3 + Z_{m_1} + \cdots + Z_{m_l} + cK_1$, where a, b, l are positive integers, c is a nonnegative integer.

Finally, we easily find that the graph G is also not the form $Z_{m_1} + Z_{m_2} + \cdots + Z_{m_l} + cK_1$ from the spectrum of Z_n, where l, c are positive integers. So G is $Z_{m_1} + \cdots + Z_{m_k}$, the largest integer (say n) in $\{m_1, \dots, m_k\}$ follows from the largest eigenvalue. Then the other m_i follows recursively by deleting Z_n from the graph and the eigenvalues of Z_n from the spectrum. $\square$

From their spectrum, we easily find the following result:

Corollary 2.9

(i) P_{2n+1} *(the path with* $2n+1$ *vertices* $)+K_1$ *is cospectral with* P_n+Z_n; *in particular*, $P_{4n+3}+2K_1$ *is cospectral with* $Z_{2n+1}+Z_n+P_n$;
(ii) $P_{2^r-1}+(r-2)K_1$ *is cospectral with* $Z_{2^{r-1}-1}+\cdots+Z_7+Z_3+P_3$, $r\geqslant 3$.

3. Graphs Z_n, W_n, T_n are determined by their Laplacian spectrum

We write the characteristic polynomial $P_{L(G)}(\mu)=|\mu I-L(G)|=q_0\mu^n+q_1\mu^{n-1}+\cdots+q_{n-1}\mu+q_n$ and summarize some results in [3,10] in the following lemma.

Lemma 3.1

(i) *Let* G *be a graph with* n *vertices and* m *edges and let* $d=(d_1,\ldots,d_n)$ *be its non-increasing degree sequence. Then some of the coefficients in* $P_{L(G)}(\mu)$ *are*:

$$q_0=1;\quad q_1=-2m;\quad q_2=2m^2-m-\frac{1}{2}\sum_{i=1}^{n}d_i^2;$$
$$q_{n-1}=(-1)^{n-1}nS(G);\quad q_n=0;$$

where m *is the number of edges of* G. $S(G)$ *is the number of spanning trees in* G.
(ii) *For the Laplacian matrix of a graph, the following follows from its spectrum*:
(a) *the number of components*.
(b) *the number of spanning trees*.

The following lemma can be found in [7,9]

Lemma 3.2. *Let* G *be a graph with* $V(G)\neq\emptyset$ *and* $E(G)\neq\emptyset$. *Then*

$$\Delta(G)+1\leqslant\mu_{\max}\leqslant\max\left\{\frac{d_u(d_u+m_u)+d_v(d_v+m_v)}{d_u+d_v},\,uv\in E(G)\right\}$$

where $\Delta(G)$ *denotes the maximum vertex degree of* G, $\mu_{\max}$ *denotes the largest Laplacian eigenvalue of* G, m_v *denotes the average of the degrees of the vertices adjacent to vertex* v *in* G.

Lemma 3.3 [6]. *Let T be a tree with n vertices and X its line graph. Then, for $i = 1, 2, \ldots, n-1$, $\mu_i(T) = \lambda_i(X) + 2$.*

Lemma 3.4 [5, Theorem 13.6.2]. *Let G' be a graph obtained by deleting an edge from the graph G. Then, for $i = 1, 2, \ldots, n-1$, $\mu_i(G) \geqslant \mu_i(G') \geqslant \mu_{i+1}(G)$.*

Lemma 3.5. *All starlike trees with the largest vertex degree 3 have the second largest Laplacian eigenvalue less than 4.*

Proof. Note each starlike tree with the largest vertex degree 3 is the disjoint union of two paths (or a path plus K_1) by deleting the edge adjacent to the vertex of degree 3 from its edge set. The Laplacian spectrum of P_n is $2 + 2\cos\frac{i\pi}{n+1}$, $i = 1, \ldots, n$. [3]. Thus $\mu_1(P_n) < 4$, by Lemma 3.4, the result follows. □

Theorem 3.6. *Graph Z_n is determined by its Laplacian spectrum.*

Proof. Suppose that a graph G and Z_n are cospectral with respect to the Laplacian matrix, then G has $n+2$ vertices. By Lemma 2.1, G and Z_n share the same characteristic polynomial of $L(G)$. So G and Z_n have the same number of edges and spanning trees by Lemma 3.1(i). Since Z_n contains one spanning tree, then G is a tree. Applying Lemma 3.2, we find that $4 \leqslant \mu_1(Z_n) \leqslant 4.4$. So G is a tree with no vertex of degree at least 4 by Lemma 3.2. At the same time, Lemma 3.1 implies

$$\sum_{i=1}^{n+2} {d_i'}^2 = \sum_{i=1}^{n+2} {d_i}^2$$

where d_i', d_i are degrees of vertex v_i in G and Z_n, respectively. It follows that G is a starlike tree with the largest vertex degree 3. Furthermore, we declare that graph G_4 (see Fig. 3) is not an induced subgraph of G. Let L' be the Laplacian matrix of G_4. By using Maple, we get the largest Laplacian eigenvalue of G_4 is about 4.414. If graph G_4 is a induced subgraph of G, then $L' + D$ is a principle submatrix of $L(G)$ for some diagonal matrix D with non-negative entries. But then $L' + D$ has the largest eigenvalue at least 4.414, a contradiction. Suppose G is non-isomorphic to Z_n, then G must be isomorphic to one of the following $\lfloor\frac{n}{2}\rfloor - 1$ graphs (Fig. 4(a)).

By Lemma 3.3, the line graph of G is cospectral with the line graph of Z_n with respect to the adjacency matrix. Therefore the line graph of G and the line graph of Z_n should have the same number of closed walks of length 4 by Lemma 2.1. But we can easily find that the numbers of closed walks of length 4 in the $\lfloor\frac{n}{2}\rfloor - 1$ line graphs (see Fig. 4(b)) are all greater than that of the line graph of Z_n (all those line graphs contain no 4 cycles, the number of induced paths of length two in the former are greater than that of the latter by 1). Hence G is isomorphic to Z_n. □

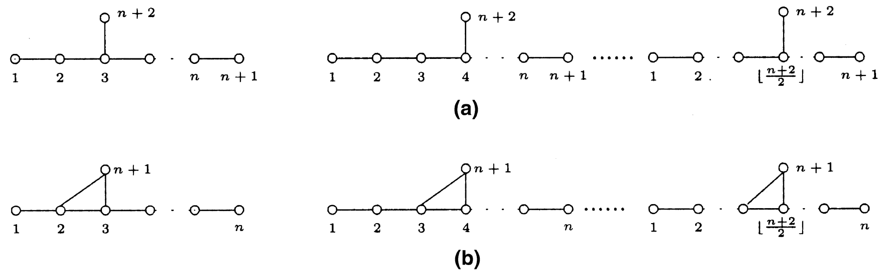

Fig. 4. $\lfloor \frac{n}{2} \rfloor - 1$ graphs and their line graphs.

Similarly, we derive:

Theorem 3.7. *Graph W_n is determined by its Laplacian spectrum.*

Proof. For $n = 1$, W_n is $K_{1,4}$. Suppose a graph X and $K_{1,4}$ are cospectral with respect to the Laplacian matrix, then X is a tree with five vertices by Lemma 3.1. But all trees with five vertices are P_5, Z_3 and $K_{1,4}$. Since P_5, Z_3 are determined by their Laplacian spectrum, respectively, so X is $K_{1,4}$. Therefore $K_{1,4}$ is determined by its Laplacian spectrum. Let $n \geqslant 2$. Suppose that G and W_n are cospectral with respect to the Laplacian matrix. Similar to the proof of Theorem 3.6, G is a tree without any vertex of degree at least 4 and exactly two vertices of degree 3. So the line graph of G and the line graph of W_n are cospectral with respect to the adjacency matrix by Lemma 3.3. Therefore they have the same number of closed walks of length 4 by Lemma 2.1. For $n = 2$, obviously, G is isomorphic to W_n. For $n = 3, n = 4$, we can easily get G isomorphic to W_n by counting the number of closed walks of length 4 in their line graphs of G and W_n, respectively. For $n \geqslant 5$. Assume that G is non-isomorphic to W_n. Similarly to the proof of Theorem 3.6, the inequalities $4 \leqslant \mu_1(W_n) = \mu_1(G) \leqslant 4.4$ hold and G_4 is not an induced subgraph of G. Then the line graph of G is one of the following graphs (Fig. 5).

Clearly, all the number of closed walk of length 4 in these graphs are greater than that of the line graph of W_n (the number of induced paths of length two in the former are all greater than that of the latter). Thus G is isomorphic to W_n. $\square$

Similarly, we obtain:

Corollary 3.8. *Graph T_n is determined by its Laplacian spectrum.*

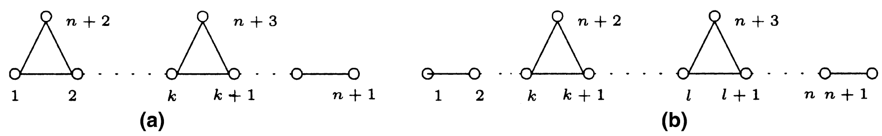

Fig. 5. The possible line graphs of G.

Proof. Similar to the proof of Theorem 3.6 and only the number of closed walks of length 6 in line graph is involved in additional. □

Theorem 3.9. *kZ_n is determined by its Laplacian spectrum.*

Proof. Suppose a graph G is cospectral with kZ_n with respect to the Laplacian matrix. Lemma 3.1 implies that graph G has $k(n+2)$ vertices, $k(n+1)$ edges and k components. So G is a forest. From the proof of Theorem 3.6, we have that $4 \leqslant \mu_1(kZ_n) = \mu_2(kZ_n) = \cdots = \mu_k(kZ_n) \leqslant 4.4$. Furthermore, by Lemma 3.5 $\mu_2(Z_n) < 4$, so $\mu_{k+1}(kZ_n) = \cdots = \mu_{2k}(kZ_n) < 4$. Similar to the proof of Theorem 3.6, G has no vertex with degree more than 3. We declare that there are exactly k vertices of degree 3 in G. Suppose that there exist x vertices of degree one, y vertices of degree two, z vertices of degree three, by Lemma 3.1 and $\sum_{v\in V(G)} d(v) = 2\varepsilon$, where ε is the number of edges in G, we then have the following equations:

$$x + y + z = k(n+2),$$

$$x + 2y + 3z = 2k(n+1),$$

$$x + 4y + 9z = 3k + 4k(n-2) + 9k.$$

Solving these equations simultaneously, we find $z = k$. Assume that there exists one path component in G, then there must exist one component with two vertices of degree 3 in G . Since the largest Laplacian eigenvalue of any path is less than 4, it forces the largest Laplacian eigenvalue and the second largest Laplacian eigenvalue are equivalent and all greater than 4 in the spectrum of one of the components except the path component in G. However, it is impossible by Lemmas 2.3, 3.3 and 3.4. Therefore each component of G contains exactly one vertex of degree 3. Furthermore, each component has the same number of vertices. Assume that there exists a component C which has $n + 2 + k$ $(k \geqslant 1)$ vertices, then $\mu_1(C) \geqslant \mu_1(Z_{n+2+k}) > Z_n$ by Lemmas 2.3, 3.3 and 3.4. Hence there exists an eigenvalue greater than $\mu_1(kZ_n)$ in the Laplacian spectrum of G, a contradiction. From the proof of Theorem 3.6, each component of G is Z_n. The result follows. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement, so the complements of graphs Z_n, W_n and T_n are determined by their Laplacian spectrum, respectively.

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